Cascading failures in coupled map lattices

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Large cascades triggered by initial shocks are common in complex networks. Coupled map lattices have been widely used over the past decades as dynamical models of complex systems. Here we investigate cascading failures in coupled map lattices with different topologies. We find that cascading failures are much easier to occur in small-world and scale-free coupled map lattices than in globally coupled map lattices.

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I. INTRODUCTION

Cascading failures have been observed in many real complex networks. The largest blackout in U.S. history took place on 14 August 2003, a typical example of cascading failure in electrical power grids. How can initial shocks lead to the entire networks to collapse and what are the features of cascading failures in different networks? Although the tendency for cascading failures in complex networks is far from completely understood, it is necessarily influenced by both the structure of a network and the dynamic behavior of network components. In particular, since the discovery of small-world and scale-free features of complex networks [1,2], some researchers have investigated the relationship between the cascading failure phenomenon and topologies of complex networks [3–7].

Coupled map lattices (CML's) have been widely investigated over the past decades to model the rich space-time dynamical behaviors of complex systems [8]. In most of these researches, a coupled map lattice is usually assumed to have a regular coupling (such as global coupling or nearest-neighbor coupling) topology. Recently, some researchers have begun to investigate dynamical behaviors such as synchronization on CML's with small-world or scale-scale coupling topologies [9].

In this work, we propose a cascading failure model based on CML's. We investigate the cascading failure in the model with different coupling topologies, including global coupling, small-world coupling, and scale-free coupling. We find that the breakdown of a single node is sufficient to trigger an entire network to collapse if the amplitude of the external perturbation on the single node is larger than a threshold. Furthermore, we find that the threshold for a globally coupled map lattice is much larger than that for a small-world or scale-free coupled map lattice. This implies that cascading failures occur much easier in small-world and scale-free networks than in global coupling networks.

II. CASCADING FAILURE MODEL BASED ON COUPLED MAP LATTICES

We consider a CML of N nodes described as follows:

$$x_i(t+1) = \left| (1-\varepsilon) f(x_i(t)) + \varepsilon \sum_{j=1,\ j\neq i}^{N} a_{i,j} f(x_j(t))/k(i) \right|, \quad i = 1,2,\ldots,N \tag{1}$$

where $x_i(t)$ is the state variable of the ith node at the tth time step. The connection information among the N nodes is given by the adjacency matrix $A=(a_{ij})_{N\times N}$. If there is an edge between node i and node j, then $a_{ij}=a_{ji}=1$; otherwise, $a_{ij}=a_{ji}=0$. Here we assume that no two different nodes can have more than one edge in between and no node can have an edge with itself. Therefore, A is a symmetric 0-1 matrix with diagonal elements zero. $k(i)$ is the degree of node i which is defined as the number of edges incident to node i. $\varepsilon \in (0,1)$ represents the coupling strength. The function f defines the local dynamics which is chosen in this work as the chaotic logistic map, $f(x)=4x(1-x)$. We use absolute value notation in Eq. (1) to guarantee that each state is always non-negative.

Node i is said to be in a *normal state* at the mth time step if $0 < x_i(t) < 1$, $t \leqslant m$. On the other hand, if $0 < x_i(t) < 1$, $t < m$; $x_i(m) \geqslant 1$, then node i is said to be *failed* at the mth time step and we assume in this case that $x_i(t) \equiv 0$, $t > m$. If the initial state of each node in network (1) is in the interval (0, 1) and there is not any external perturbation, then N nodes in the network will be in normal states forever.

In order to show how an initial shock on a single node can trigger cascading failure, we add an external perturbation $R \geqslant 1$ to a node c at the mth time step as follows:

$$x_c(m) = \left| (1-\varepsilon) f(x_c(m-1)) + \varepsilon \sum_{j=1\&j\neq c}^{N} a_{c,j} f(x_j(m-1))/k(c) \right| + R. \tag{2}$$

In this case, node c will be failed at the mth time step and we have $x_c(t) \equiv 0$ for all $t > m$. At the $(m+1)$th time step, the states of those nodes that are directly connected with node c will be affected by $x_c(m)$ according to Eq. (1), and the states of these nodes may also be larger than 1 and thus may lead to a new round of node failures. The question we are interested in is how many nodes will be failed eventually?

In following simulations, initial states of the nodes in coupled map lattice (1) are all chosen randomly from the interval (0, 1). A perturbation $R \geqslant 1$ is added to a node c at the tenth time step. The cascading failure process can be characterized by $I(t)$ which is defined as the total number of

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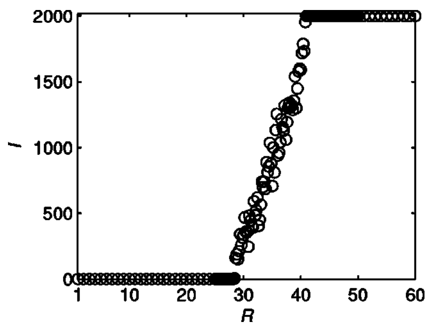

FIG. 1. The size I of cascade in a global coupled map lattice with $N=2000$ and $\varepsilon=0.6$, triggered by adding an initial shock R to an arbitrarily chosen node. The data are averages over 100 random realizations.

failed nodes in a network before the $(t+1)$th time step. $I \equiv \lim_{t\to\infty} I(t)$ measures the size of cascade in the network.

III. CASCADING FAILURE IN GLOBALLY COUPLED MAP LATTICES

In a globally coupled network, each node is connected with all the other nodes in the network. Here, a perturbation $R \geqslant 1$ is added to a randomly selected node in a globally coupled map lattice (1) at the tenth time step. Figure 1 shows the size I of the cascade as a function of the amplitude R of a perturbation in a globally coupled map lattice with $N=2000$ and $\varepsilon=0.6$. We find that for any given size N of the network and coupling strength $\varepsilon \in (0,1)$, there exist two thresholds $R_{c1} \equiv R_{c1}(\varepsilon, N)$ and $R_{c2} \equiv R_{c2}(\varepsilon, N)$ $(R_{c1} < R_{c2})$, as shown in Fig. 2. Below threshold R_{c1}—i.e., $1 < R \leqslant R_{c1}$—$I=1$, which implies that no other nodes will be failed. However, as R increases from R_{c1}, the size I of the cascade increases very rapidly. Once the amplitude R of the perturbation reaches another threshold R_{c2}—i.e., $R \geqslant R_{c2}$—then all other nodes in the network will be failed in the next time step $(I \equiv N)$. In Fig. 1, the two thresholds are $R_{c1} \approx 29.4$ and $R_{c2} \approx 41.4$.

Mathematically, the threshold R_{c2} can be estimated as follows. All other nodes failed in the $(m+1)$th time step means that

$$x_i(m+1) = \left| f(x_i(m)) + \frac{\varepsilon}{N-1}\sum_{j=1}^{N}[f(x_j(m)) - f(x_i(m))] \right| \geqslant 1,$$

$$i \neq c \tag{3}$$

Note that

$$f(x_c(m)) = 4x_c(1-x_c) \leqslant -4R(R-1) \leqslant 0. \tag{4}$$

Denote $\overline{x_c}(m)$ as the state of node c at the mth time step without external perturbation R. We have $0 \leqslant \overline{x_c}(m) \leqslant 1$. It can be derived that

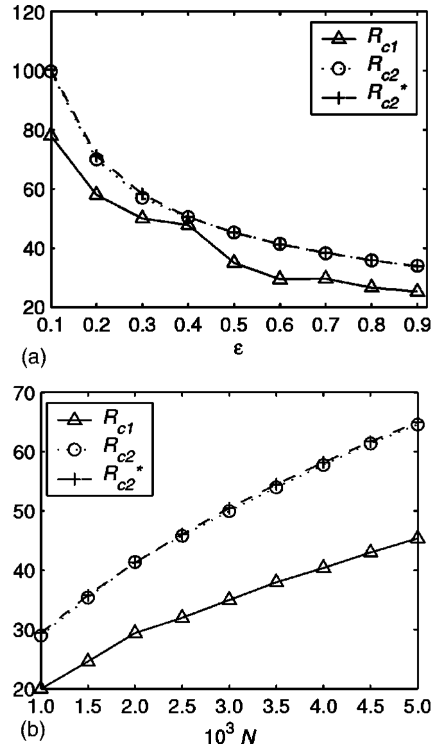

FIG. 2. Thresholds for cascading failures in globally coupled map lattices with $N=2000$ and $\varepsilon=0.6$. R_{c1} and R_{c2} are derived from the data averages over 100 random realizations. R_{c2}^* is computed according to Eq. (6). Below threshold R_{c1}, cascading failure will not occur. Above threshold R_{c2} (R_{c2}^*), all the nodes in the network will be failed. The thresholds are decreasing functions of the coupling strength ε (a) and increasing functions of network size N (b).

$$\begin{aligned}
f(x_i(m)) &+ \frac{\varepsilon}{N-1}\sum_{j=1}^{N}[f(x_j(m)) - f(x_i(m))] \\
&= f(x_i(m)) + \frac{\varepsilon}{N-1}\Bigg\{\sum_{\substack{j=1\\ j\neq i}}^{N}[f(x_j(m)) - f(x_i(m))] \\
&\quad + [f(\overline{x_c}(m)) - f(x_i(m))] + [f(x_c(m)) - f(\overline{x_c}(m))]\Bigg\} \\
&\leqslant 1 + \frac{\varepsilon}{N-1}[f(x_c(m)) - f(\overline{x_c}(m))] \\
&\leqslant 1 + \frac{\varepsilon}{N-1} f(x_c(m)).
\end{aligned} \tag{5}$$

Therefore, Eq. (3) holds if

$$1 + \frac{\varepsilon}{N-1} f(x_c(m)) \leqslant 1 - \frac{4\varepsilon R(R-1)}{N-1} \leqslant -1,$$

which leads to

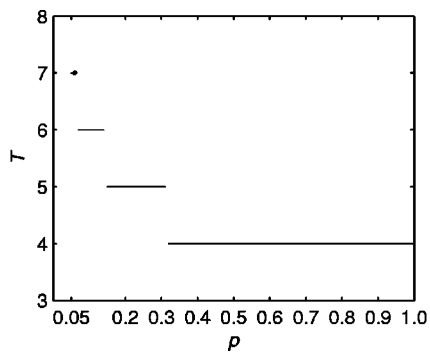

FIG. 3. The number T of time steps required to achieve global cascading failure in a small-world coupled map lattice as a function of the rewiring probability p.

$$R \geqslant R_{c2}^{*} \equiv \frac{1}{2}\left(1 + \sqrt{1 + \frac{2(N-1)}{\varepsilon}}\right). \tag{6}$$

It can be seen from Fig. 2 that the threshold R_{c2}^{*} estimated from Eq. (6) fits very well with the threshold R_{c2} derived from simulations.

It should be noted that in the thermodynamic limit $N \to \infty$, thresholds R_{c1} and R_{c2} increase to infinity [Fig. 2(b)]. This implies that a large-scale globally coupled map lattice is very robust against a large but local external perturbation. Although globally coupled network models capture some important features of real networks, it is easy to notice their limitations: a globally coupled network with N nodes has $N(N-1)/2$ edges, while most large-scale real networks are sparse; that is, the number of edges in a real network is generally of order N rather than N^2.

IV. CASCADING FAILURE IN SMALL-WORLD COUPLED MAP LATTICES

A widely studied, sparse, and regular network model is the nearest-neighbor coupled network which consists of N nodes arranged in a ring, where each node i is adjacent to its neighboring nodes, $i=1,2,\ldots,K/2$, with K being an even integer. In simulations, we take $K=20$, $\varepsilon=0.6$, and $N \geqslant 1000$. A perturbation $R \geqslant 1$ is added to a random selected node at the mth time step. We find that for a large perturbation $(R>6)$, the number of failed nodes before the $(m+t+1)$th time step is about $I(m+t)=Kt+1$. Therefore, it requires about N/K time steps for the cascading failure of all the nodes in a nearest-neighbor coupled map lattice and $N/K \to \infty$ as $N \to \infty$. However, in simulations, we cannot produce the whole cascading process due to the occurrence of arithmetic overflow.

Many real networks have special features, which are a blend of those of completely regular networks and completely random networks. To describe the transition from a completely regular network to a completely random one, Watts and Strogatz introduced the small-world network model [1]. Starting from a nearest-neighbor coupled network with N nodes arranged on a ring and K edges per node, they rewire each edge at random with probability p. Watts and Strogatz (WS) quantified the structural properties of these

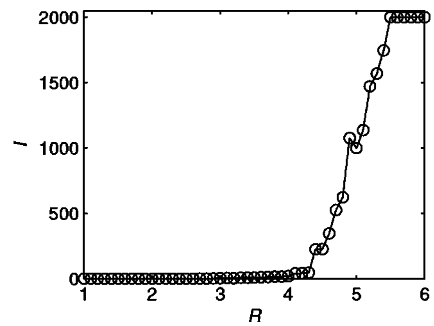

FIG. 4. The size I of cascade in a small-world coupled map lattice with $N=2000$, $K=20$, $p=0.05$, and $\varepsilon=0.6$, triggered by adding an initial shock R to an arbitrarily chosen node. The data are averages over 100 random realizations.

networks by their characteristic path length $L(p)$ and clustering coefficient $C(p)$. $L(p)$ measures the typical separation between two nodes (a global property) and $C(p)$ measures the cliquishness of a typical neighborhood (a local property). They found that, for small p $(0<p\ll 1)$, $L(p)$ drops rapidly while $C(p)$ remains almost unchanged. The ensuing semirandom lattice is denoted a small-world network. In a WS small-world network, most nodes only connect to their nearest-neighbor nodes but a few nodes have long-range connections with relatively distant nodes. The total number of long-range connections in the network is subject to $pNK/2$.

We find that it is much easier to trigger global cascading failure in a small-world coupled map lattice than in a nearest-neighbor coupled map lattice. In simulations, we take $N=2000$, $K=20$, $R=6$, and $\varepsilon=0.6$. In Fig. 3 we plot the number T of time steps required to achieve global cascading failure in a small-world coupled map lattice as a function of the rewiring probability p. We find that $T \leqslant 7$ for $p \geqslant 0.05$ and $T \equiv 4$ for $0.32 \leqslant p \leqslant 1$. Figure 4 plots size I of the cascade as a function of the amplitude R of perturbation with $p=0.05$, when a randomly selected node is failed at the tenth time step due to the perturbation. We find that all the nodes in the network will be failed in a few steps if $R>5.5$. For example, for $R=6$, all the nodes in the network are failed after 7 steps (Fig. 5). This implies that a few long-range connections are enough for a single node failure to trigger large-scale network collapse in a few steps.

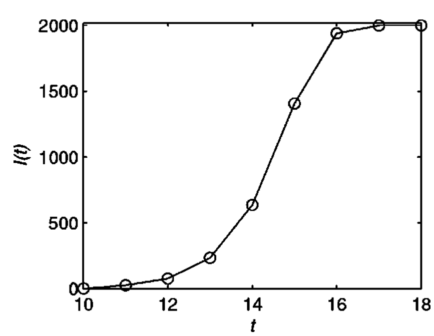

FIG. 5. Cascading failure process in a small-world coupled map lattice with $N=2000$, $K=20$, $\varepsilon=0.6$, $p=0.05$, and $R=6$, triggered by adding an initial shock R to an arbitrarily chosen node at tenth step. All the nodes in the network are failed after 7 steps.

V. CASCADING FAILURE IN SCALE-FREE COUPLED MAP LATTICES

One significant recent discovery in the field of complex networks is the observation that the connectivity distributions of a number of large-scale and complex networks have the power-law form $P(k) \sim k^{-\gamma}$, where $P(k)$ is the probability that a node in the network is connected to k other nodes and γ is a positive real number. Since power laws are free of characteristic scale, such networks are called "scale-free networks." Barabási and Albert (BA) argued that there are two generic aspects of real networks in the scale-free structure model, which are growth and preferential attachment [2]. They have referred to that network continuously grown by the addition of new vertices and new vertices are preferentially attached to existing vertices with high numbers of connections. The BA scale-free model is constructed as follows [2].

(i) *Growth:* Starting with a small number (m_0) of nodes, at every time step we add a new node with $m(\leq m_0)$ edges.

(ii) *Preferential attachment:* When choosing the nodes to which the new node connects, we assume that the probability Π that a new node will be connected to a node depends on the connectivity of that node, such that

$$\Pi(k_i) = k_i \Big/ \sum_j k_j .$$

After l time steps the model leads to a scale-free network with $N=l+m_0$ nodes and ml edges. In simulations, we take $m_0=m$.

A scale-free network is inhomogeneous in nature: most nodes have very few connections but a small number of particular nodes have many connections. It is this inhomogeneous feature that makes the connectivity of a scale-free network error tolerant but vulnerable to deliberate attacks [10]. More precisely, the connectivity of such networks is highly robust against random failures—that is, random removal of nodes; yet it is extremely fragile to attacks, that is, to specific removal of the most highly connected nodes.

We investigate cascading failures in BA scale-free coupled map lattices with $N=2000$ and $\varepsilon=0.6$. To take into account the inhomogeneous feature of a scale-free network, we adopt two different triggering strategies: random attack and deliberate (degree-based) attack. In the random attack case, an initial shock is added to a randomly chosen node. In the deliberate (degree-based) attack case, an initial shock is added to the node with largest degree in the network. Figures 6(a) and 6(b) plot the size of cascade in a scale-free coupled map lattice as a function of the amplitude of the perturbation under random attack and deliberate attack, respectively. One can see that there is no sharp difference between random and deliberate attacks. In each case, there exists a similar threshold R^*_{BA}. Below the threshold, at most a few nodes (less than 10 in our simulations) will be failed. As the value of R increases from the threshold, the size I of cascade increases very sharply to the size N of the network. Furthermore, the threshold R^*_{BA} is much smaller than the threshold R_{c1} for a globally coupled map lattice with the same size N and coupling strength ε. This implies that it is much easier to trigger network collapse in a scale-free network than in a globally coupled network. Figure 7 shows the cascading failure process in a scale-free coupled map lattice with $m=3$ and $R=10$.

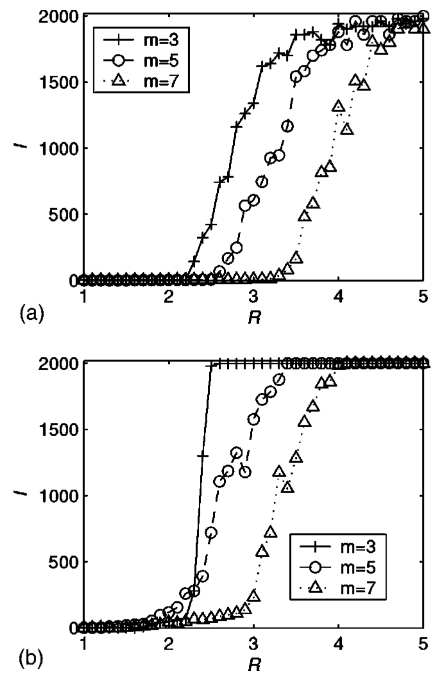

FIG. 6. The size of cascade in a scale-free coupled map lattice, triggered by adding an initial shock to an arbitrarily chosen node (a) or a node with largest degree (b). The data are averages over 100 random realizations.

The above results about cascading failures on the BA scale-free networks are consistent with the recent discoveries of Fortunato [11], who studied damage spreading for the Krause-Hegselmann opinion dynamics on BA scale-free networks. Fortunato distinguished three phases in the confidence bound space, corresponding to zero, partial, and total damage, respectively. Furthermore, he found that the amount of damage depends on the degree of the damaged node but the thresholds for damage spreading and saturation do not, because of the small-world effect of BA scale-free networks [11].

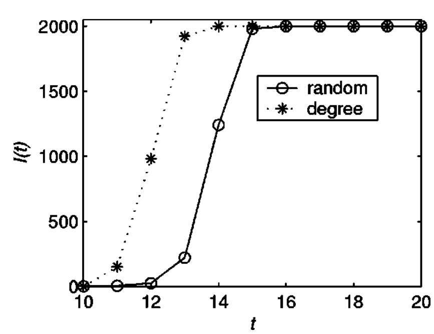

FIG. 7. Cascading failure process in a scale free coupled map lattice with $N=2000$, $\varepsilon=0.6$, $m=3$, and $R=10$, triggered by adding an initial shock R to an arbitrarily chosen node or a node with largest degree at tenth step.

VI. CONCLUSION

In this work, we studied cascading failures in coupled map lattices with different coupling topologies, including global coupling, nearest-neighbor coupling, small-world coupling, and scale-free coupling. We found that a sufficiently large perturbation on a single node can lead to cascading failure of all the other nodes in the network. The perturbation threshold for the occurrence of cascading failure in a globally coupled map lattice tends to infinity in the thermodynamic limit case. On the other hand, even a small perturbation may trigger large-scale cascading failure in a small-world or scale-free coupled map lattice in a few steps. We hope this work might shed some new light on the analysis and control of cascading failures in real-world complex networks.

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